

# ON SOME MULTIPLICITY AND MIXED MULTIPLICITY FORMULAS <sup>1</sup>

Duong Quoc Viet and Truong Thi Hong Thanh

Department of Mathematics, Hanoi National University of Education

136 Xuan Thuy Street, Hanoi, Vietnam

Emails: duongquocviet@fmail.vnn.vn and thanhth@hnue.edu.vn

**ABSTRACT:** This paper gives the additivity and reduction formulas for mixed multiplicities of multi-graded modules  $M$  and mixed multiplicities of arbitrary ideals, and establishes the recursion formulas for the sum of all the mixed multiplicities of  $M$ . As an application of these formulas we get the recursion formulas for the multiplicity of multi-graded Rees modules.

## 1. Introduction

Let  $(A, \mathfrak{m})$  be an artinian local ring with maximal ideal  $\mathfrak{m}$ , infinite residue field  $k = A/\mathfrak{m}$ . Let  $S = \bigoplus_{n_1, \dots, n_d \geq 0} S_{(n_1, \dots, n_d)}$  ( $d > 0$ ) be a finitely generated standard  $\mathbb{N}^d$ -graded algebra over  $A$  (i.e.,  $S$  is generated over  $A$  by elements of total degree 1) and let  $M = \bigoplus_{n_1, \dots, n_d \geq 0} M_{(n_1, \dots, n_d)}$  be a finitely generated  $\mathbb{N}^d$ -graded  $S$ -module such that  $M_{(n_1, \dots, n_d)} = S_{(n_1, \dots, n_d)} M_{(0, \dots, 0)}$  for all  $n_1, \dots, n_d \geq 0$ . Throughout this paper, put  $S_i = S_{(0, \dots, \underbrace{1}_i, \dots, 0)}$  for all  $i = 1, \dots, d$  and

$$\begin{aligned} S^\Delta &= \bigoplus_{n \geq 0} S_{(n, \dots, n)}, \quad S_{++} = \bigoplus_{n_1, \dots, n_d > 0} S_{(n_1, \dots, n_d)}, \\ S_+^\Delta &= \bigoplus_{n > 0} S_{(n, \dots, n)}, \quad S_+ = \bigoplus_{n_1 + \dots + n_d > 0} S_{(n_1, \dots, n_d)}, \\ M^\Delta &= \bigoplus_{n \geq 0} M_{(n, \dots, n)}, \quad \mathfrak{a} : \mathfrak{b}^\infty = \bigcup_{n \geq 0} (\mathfrak{a} : \mathfrak{b}^n). \end{aligned}$$

Denote by  $\text{Proj } S$  the set of the homogeneous prime ideals of  $S$  which do not contain  $S_{++}$ . Set  $\dim M^\Delta = \ell$  and

$$\text{Supp}_{++} M = \left\{ P \in \text{Proj } S \mid M_P \neq 0 \right\}.$$

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This research was in part supported by a grant from NAFOSTED.

*Mathematics Subject Classification* (2010): Primary 13H15. Secondary 13A02, 13A15, 13A30, 14C17.

*Key words and phrases* : Noetherian ring, mixed multiplicity, multi-graded module, filter-regular sequence.

By [24, Remark 3.1],  $\dim \operatorname{Supp}_{++} M = \ell - 1$ . And by [4, Theorem 4.1],  $\ell_A[M_{(n_1, \dots, n_d)}]$  is a polynomial of degree  $\ell - 1$  for all large  $n_1, \dots, n_d$ . The terms of total degree  $\ell - 1$  in this polynomial have the form

$$\sum_{k_1 + \dots + k_d = \ell - 1} e(M; k_1, \dots, k_d) \frac{n_1^{k_1} \dots n_d^{k_d}}{k_1! \dots k_d!}.$$

Then  $e(M; k_1, \dots, k_d)$  are called the *mixed multiplicity of type  $(k_1, \dots, k_d)$  of  $M$*  [4]. In the case that  $(R, \mathfrak{n})$  is a noetherian local ring with maximal ideal  $\mathfrak{n}$ ,  $J$  is an  $\mathfrak{n}$ -primary ideal,  $I_1, \dots, I_d$  are ideals of  $R$ ,  $N$  is a finitely generated  $R$ -module, then it is easily seen that

$$F_J(J, I_1, \dots, I_d; N) = \bigoplus_{n_0, n_1, \dots, n_d \geq 0} \frac{J^{n_0} I_1^{n_1} \dots I_d^{n_d} N}{J^{n_0+1} I_1^{n_1} \dots I_d^{n_d} N}$$

is a finitely generated graded  $F_J(J, I_1, \dots, I_d; R)$ -module. Mixed multiplicities of  $F_J(J, I_1, \dots, I_d; N)$  are denoted by  $e(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_d^{[k_d]}, N)$  and which are called *mixed multiplicities of  $N$  with respect to ideals  $J, I_1, \dots, I_d$*  (see [11, 20]).

Although the problems of expressing the multiplicity of graded modules in terms of mixed multiplicities and the relationship between mixed multiplicities and Hilbert-Samuel multiplicity have attracted much attention in past years (the citations will be mentioned in the next sections), the properties similar to that of the Hilbert-Samuel multiplicity (for instance: the additive property on exact sequences as in [5, Lemma 17.4.4] and the additivity and reduction formula [5, Theorem 17.4.8] for mixed multiplicities of  $\mathfrak{n}$ -primary ideals...) for mixed multiplicities of arbitrary ideals and multi-graded modules, and other properties, are not yet known.

In the present paper, by a new approach we give additivity and reduction formulas for mixed multiplicities of multi-graded modules and mixed multiplicities of arbitrary ideals. And we establish the recursion formulas for the sum of all the mixed multiplicities of multi-graded modules.

As one might expect, we first obtain the following result for mixed multiplicities of multi-graded modules.

**Theorem 3.1.** *Let  $S$  be a finitely generated standard  $\mathbb{N}^d$ -graded algebra over an artinian local ring  $A$  and let  $M$  be a finitely generated  $\mathbb{N}^d$ -graded  $S$ -module such that  $S_{(1,1,\dots,1)}$  is not contained in  $\sqrt{\operatorname{Ann} M}$ . Denote by  $\Lambda$  the set of all homogeneous prime ideals  $P$  of  $S$  such that  $P \in \operatorname{Supp}_{++} M$  and  $\dim \operatorname{Proj}(S/P) = \dim \operatorname{Supp}_{++} M$ . Then*

$$e(M; k_1, \dots, k_d) = \sum_{P \in \Lambda} \ell(M_P) e(S/P; k_1, \dots, k_d).$$

We would like to emphasize that although Theorem 3.1 is a general result for mixed multiplicities of multi-graded modules which is a general object for mixed multiplicities of ideals, up to now we can not prove the following theorem by using Theorem 3.1.

**Theorem 3.2.** *Let  $(R, \mathfrak{n})$  be a noetherian local ring with maximal ideal  $\mathfrak{n}$ , infinite residue field  $k = R/\mathfrak{n}$ , ideals  $I_1, \dots, I_d$ , an  $\mathfrak{n}$ -primary  $J$ . Let  $N$  be a finitely generated  $R$ -module. Assume that  $I = I_1 \cdots I_d$  is not contained in  $\sqrt{\text{Ann} N}$ . Set  $\overline{N} = \frac{N}{0_N : I^\infty}$ . Denote by  $\Pi$  the set of all prime ideals  $\mathfrak{p}$  of  $R$  such that  $\mathfrak{p} \in \text{Min}(R/\text{Ann} \overline{N})$  and  $\dim R/\mathfrak{p} = \dim \overline{N}$ . Then we have*

$$e(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_d^{[k_d]}; N) = \sum_{\mathfrak{p} \in \Pi} \ell(N_{\mathfrak{p}}) e(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_d^{[k_d]}; R/\mathfrak{p}).$$

It is natural to suppose that the proof of Theorem 3.2 will have to use the additive property on exact sequences of mixed multiplicities. But in fact, this approach seems to become a obstruction in proving Theorem 3.2. This is a motivation to help us giving another approach for the proof of Theorem 3.2 as in this paper (see the proof of Theorem 3.2, Section 3). On the contrary, even from Theorem 3.2 we show that mixed multiplicities of arbitrary ideals are additive on exact sequences (see Corollary 3.9, Section 3) which covers [5, Lemma 17.4.4].

Our approach is based on multiplicity formulas of multi-graded Rees modules with respect to powers of ideals (see Proposition 2.4 and Corollary 2.5, Section 2) via linking minimal homogeneous prime ideals of maximal coheight of the Rees module  $\mathfrak{R}(I_1, \dots, I_d; N) = \bigoplus_{n_1, \dots, n_d \geq 0} I_1^{n_1} \cdots I_d^{n_d} N$  and minimal prime ideals of maximal coheight of  $N$  (see Lemma 3.4, Section 3).

Set

$$S_{\hat{i}} \bigoplus_{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d \geq 0; n_i=0} S_{(n_1, \dots, n_d)} \text{ and } M_{\hat{i}} = S_{\hat{i}} M_{(0, \dots, 0)}.$$

Next, we establish the recursion formulas for the sum of all the mixed multiplicities of the  $\mathbb{N}^d$ -graded module  $M : \tilde{e}(M) = \sum_{k_1 + \dots + k_d = \ell-1} e(M; k_1, \dots, k_d)$  which express  $\tilde{e}(M)$  as a sum

$$\tilde{e}(M) = \tilde{e}(M/xM) + \tilde{e}(W),$$

where

$$\dim \text{Supp}_{++}(M/xM) = \dim \text{Supp}_{++} M - 1$$

and  $W$  is an  $\mathbb{N}^{d-1}$ -graded module. This result can be stated as follows.

**Theorem 5.2.** *Let  $S$  be a finitely generated standard  $\mathbb{N}^d$ -graded algebra over an artinian local ring  $A$  and let  $M$  be a finitely generated  $\mathbb{N}^d$ -graded  $S$ -module such that  $M = SM_{(0,\dots,0)}$ . Set  $\dim_{S^\Delta} M^\Delta = \ell$ . Assume that  $e(M; k_1, \dots, k_d) \neq 0$  and  $k_i > 0$ . Let  $x \in S_i$  be an  $S_{++}$ -filter-regular element with respect to  $M$ . Set  $\mathbf{h} = h_1, \dots, h_d$  and  $|\mathbf{h}| = h_1 + \dots + h_d$ . Then the following statements hold.*

- (i)  $\tilde{e}\left(\frac{M}{xM}\right) = \sum_{|\mathbf{h}|=\ell-1; h_i>0} e(M; \mathbf{h})$ .
- (ii)  $\sum_{|\mathbf{h}|=\ell-1; h_i=0} e(M; \mathbf{h}) \neq 0$  if and only if  $\dim_{S_i^\Delta} [S_i^v M_i]^\Delta = \ell$  for some  $v \gg 0$ . In this case,  $\tilde{e}(S_i^v M_i) = \sum_{|\mathbf{h}|=\ell-1; h_i=0} e(M; \mathbf{h})$  for all  $v \gg 0$ .
- (iii) If  $\dim_{S_i^\Delta} [S_i^v M_i]^\Delta = \ell$  for some  $v \gg 0$  then  $\tilde{e}(M) = \tilde{e}\left(\frac{M}{xM}\right) + \tilde{e}(S_i^v M_i)$  for all  $v \gg 0$ .
- (iv) If  $\dim_{S_i^\Delta} [S_i^v M_i]^\Delta < \ell$  for some  $v \gg 0$  then  $\tilde{e}(M) = \tilde{e}\left(\frac{M}{xM}\right)$  for all  $v \gg 0$ .

As consequences of Theorem 5.2, we get the recursion formulas for the multiplicity of multi-graded Rees modules (see Theorem 5.5; Corollary 5.6; Corollary 5.7 and Corollary 5.8, Section 5).

The main results of this paper yield many interesting consequences such as the additivity and reduction formulas for mixed multiplicities of ideals of positive height that covers [5, Theorem 17.4.8] for the case of  $\mathfrak{n}$ -primary ideals; the additive property on exact sequences for mixed multiplicities of ideals and the multiplicity of multi-graded Rees modules; the recursion formulas for the multiplicity of multi-graded Rees modules; and the multiplicity formulas of Rees modules.

This paper is divided into five sections. Section 2 is devoted to the discussion of mixed multiplicities of multi-graded Rees modules and the multiplicity of Rees modules with respect to powers of ideals (Proposition 2.4 and Corollary 2.5) that will be used as a tool in the proofs of the paper. Section 3 gives the additivity and reduction formulas for mixed multiplicities of multi-graded modules and mixed multiplicities of arbitrary ideals. Section 4 investigates the relationship between filter-regular sequences of multi-graded  $F_J(J, I_1, \dots, I_d; R)$ -module  $F_J(J, I_1, \dots, I_d; N)$  and weak-(FC)-sequences of ideals that will be used in the proofs of Section 5 (Proposition 4.5). Section 5 introduces the recursion formulas for the sum of all the mixed multiplicities of multi-graded modules. And as an application, we obtain the recursion formulas for the multiplicity of multi-graded Rees modules.

## 2. Multiplicity of multi-graded Rees modules

This section studies mixed multiplicities and the multiplicity of multi-graded modules. We will give multiplicity and mixed multiplicity formulas of Rees modules with respect to powers of ideals that will be used as a tool in the proofs of the paper.

Set  $\dim M^\Delta = \ell$ . By [4, Theorem 4.1],  $\ell_A[M_{(n_1, \dots, n_d)}]$  is a polynomial of degree  $\dim \text{Supp}_{++} M$  for all large  $n_1, \dots, n_d$ . Remember that  $\dim \text{Supp}_{++} M = \ell - 1$  by [24, Remark 3.1]. The terms of total degree  $\ell - 1$  in this polynomial have the form

$$B_M(n_1, n_2, \dots, n_d) = \sum_{k_1 + \dots + k_d = \ell - 1} e(M; k_1, \dots, k_d) \frac{n_1^{k_1} \dots n_d^{k_d}}{k_1! \dots k_d!}.$$

Then  $e(M; k_1, \dots, k_d)$  are non-negative integers not all zero, called the *mixed multiplicity of type  $(k_1, \dots, k_d)$  of  $M$*  [4]. And from now on  $B_M(n_1, n_2, \dots, n_d)$  is called the *Bahattacharya homogeneous polynomial* of  $M$  [1].

Set  $\mathbf{k} = k_1, \dots, k_d$  and  $|\mathbf{k}| = k_1 + \dots + k_d$ . Denote by  $\tilde{e}(M)$  the sum of all the mixed multiplicities of  $M$ , i.e.,  $\tilde{e}(M) := \sum_{|\mathbf{k}| = \ell - 1} e(M; \mathbf{k})$ . It is well known that in generally, the multiplicity  $e(M)$  of  $M$  and  $\tilde{e}(M)$  are different invariants of  $M$ .

Let  $(R, \mathfrak{n})$  be a noetherian local ring with maximal ideal  $\mathfrak{n}$ , infinite residue field  $k = R/\mathfrak{n}$  and let  $N$  be a finitely generated  $R$ -module. Let  $I_1, \dots, I_d$  be ideals of  $R$  such that  $I_1 \dots I_d$  is not contained in  $\sqrt{\text{Ann} N}$ .

Put  $\mathbf{I} = I_1, \dots, I_d$ ;  $\mathbf{n} = n_1, \dots, n_d$ ;  $\mathbb{I}^{\mathbf{n}} = I_1^{n_1}, \dots, I_d^{n_d}$ ;  $\mathbf{I}^{[\mathbf{k}]} = I_1^{[k_1]}, \dots, I_d^{[k_d]}$ .

Denote by

$$\mathfrak{R}(\mathbf{I}; R) = \mathfrak{R}(I_1, \dots, I_d; R) = \bigoplus_{n_1, \dots, n_d \geq 0} I_1^{n_1} \dots I_d^{n_d}$$

the Rees algebra of ideals  $I_1, \dots, I_d$  and by

$$\mathfrak{R}(\mathbf{I}; N) = \mathfrak{R}(I_1, \dots, I_d; N) = \bigoplus_{n_1, \dots, n_d \geq 0} I_1^{n_1} \dots I_d^{n_d} N$$

the Rees module of ideals  $I_1, \dots, I_d$  with respect to  $N$ . Let  $J$  be an  $\mathfrak{n}$ -primary ideal. Set

$$F_J(J, \mathbf{I}; R) = F_J(J, I_1, \dots, I_d; R) = \bigoplus_{n_0, n_1, \dots, n_d \geq 0} \frac{J^{n_0} I_1^{n_1} \dots I_d^{n_d}}{J^{n_0+1} I_1^{n_1} \dots I_d^{n_d}}$$

and

$$F_J(J, \mathbf{I}; N) = F_J(J, I_1, \dots, I_d; N) = \bigoplus_{n_0, n_1, \dots, n_d \geq 0} \frac{J^{n_0} I_1^{n_1} \dots I_d^{n_d} N}{J^{n_0+1} I_1^{n_1} \dots I_d^{n_d} N}.$$

Then  $F_J(J, \mathbf{I}; R)$  is a finitely generated standard multi-graded algebra over an artinian local ring  $R/J$  and  $F_J(J, \mathbf{I}; N)$  is a finitely generated multi-graded  $F_J(J, \mathbf{I}; R)$ -module. Set  $I = I_1 \cdots I_d$ . Denote by  $B_N(J, \mathbf{I}; n_0, \mathbf{n}) = B_N(J, \mathbf{I}; n_0, n_1, \dots, n_d)$  the Bahattacharya homogeneous polynomial of  $F_J(J, \mathbf{I}; N)$ . Then remember that

$$\deg B_N(J, \mathbf{I}; n_0, \mathbf{n}) = \dim \frac{N}{0_N : I^\infty} - 1$$

by [21, Proposition 3.1] (see [11, Proposition 3.1]). And by [24, Remark 3.1],  $\deg B_N(J, \mathbf{I}; n_0, \mathbf{n}) = \dim F_J(J, \mathbf{I}; N)^\Delta - 1$ . Hence  $\dim F_J(J, \mathbf{I}; N)^\Delta = \dim \frac{N}{0_N : I^\infty}$ .

In the case that  $\text{ht} \frac{I + \text{Ann} N}{\text{Ann} N} > 0$ ,  $\dim \frac{N}{0_N : I^\infty} = \dim N$ . The above facts yield:

**Note 2.1.**  $\dim F_J(J, \mathbf{I}; N)^\Delta = \dim \frac{N}{0_N : I^\infty}$ , and if  $\text{ht} \frac{I + \text{Ann} N}{\text{Ann} N} > 0$  then

$$\dim F_J(J, \mathbf{I}; N)^\Delta = \dim N.$$

Set  $\dim \frac{N}{0_N : I^\infty} = q$  and

$$e(F_J(J, \mathbf{I}; N); k_0, k_1, \dots, k_d) = e(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_d^{[k_d]}; N) := e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$$

( $k_0 + k_1 + \dots + k_d = k_0 + |\mathbf{k}| = q - 1$ ). Then  $e(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_d^{[k_d]}; N)$  is called the *mixed multiplicity of  $N$  with respect to ideals  $J, I_1, \dots, I_d$  of type  $(k_0, k_1, \dots, k_d)$*  (see [11, 20]).

**Note 2.2.** Recall that by [11, Proposition 3.1] which is a generalized result of [21, Proposition 3.1], we have  $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e\left(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; \frac{N}{0_N : I^\infty}\right)$ , and hence

$$\tilde{e}(F_J(J, \mathbf{I}; N)) = \tilde{e}\left(F_J(J, \mathbf{I}; \frac{N}{0_N : I^\infty})\right).$$

**Note 2.3.** By [4, Corollary 4.6], it follows that

$$e((J, \mathfrak{R}(\mathbf{I}; R)_+); \mathfrak{R}(\mathbf{I}; N)) = e(F_J(J, \mathbf{I}; N)).$$

Now, assume that  $\text{ht} \frac{I + \text{Ann} N}{\text{Ann} N} > 0$ . Then  $\dim \frac{N}{0_N : I^\infty} = \dim N$ . In this case,

$$B_N(J, \mathbf{I}; n_0, \mathbf{n}) = \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \frac{n_0^{k_0} n_1^{k_1} \cdots n_d^{k_d}}{k_0! k_1! \cdots k_d!} \quad (1)$$

and

$$e((J, \mathfrak{R}(\mathbf{I}; R)_+); \mathfrak{R}(\mathbf{I}; N)) = \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \quad (2)$$

by [4, Theorem 4.4] which is a generalized version of [20, Theorem 1.4]. Next, let  $u_1, \dots, u_d$  be positive integers. Set  $\mathbf{u}^{\mathbf{k}} = u_1^{k_1} \dots u_d^{k_d}$ . From (1) we have

$$B_N(J, \mathbf{I}^{\mathbf{u}}, n_0, \mathbf{n}) = \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{\mathbf{u}[\mathbf{k}]}; N) \frac{n_0^{k_0} n_1^{k_1} \dots n_d^{k_d}}{k_0! k_1! \dots k_d!} \quad \text{and}$$

$$B_N(J, \mathbf{I}^{\mathbf{u}}, n_0, \mathbf{n}) = \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \frac{n_0^{k_0} (u_1 n_1)^{k_1} \dots (u_d n_d)^{k_d}}{k_0! k_1! \dots k_d!}.$$

Consequently,  $e(J^{[k_0+1]}, \mathbf{I}^{\mathbf{u}[\mathbf{k}]}; N) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \mathbf{u}^{\mathbf{k}}$ . Hence by (2),

$$e((J, \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R)_+); \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; N)) = \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \mathbf{u}^{\mathbf{k}}.$$

We obtain the following result.

**Proposition 2.4.** *Assume that  $\text{ht} \frac{I + \text{Ann} N}{\text{Ann} N} > 0$  and  $u_1, \dots, u_d$  are positive integers. Then*

- (i)  $e(J^{[k_0+1]}, \mathbf{I}^{\mathbf{u}[\mathbf{k}]}; N) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \mathbf{u}^{\mathbf{k}}$ .
- (ii)  $e((J, \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R)_+); \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; N)) = \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \mathbf{u}^{\mathbf{k}}$ .

Set  $\overline{N} = \frac{N}{0_N : I^\infty}$ . It can be verified that  $\text{ht} \frac{I + \text{Ann} \overline{N}}{\text{Ann} \overline{N}} > 0$ . By Note 2.2,

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e\left(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; \frac{N}{0_N : I^\infty}\right).$$

Then as an immediate consequence of Proposition 2.4 we get the following.

**Corollary 2.5.** *Let  $u_1, \dots, u_d$  be positive integers. Then*

- (i)  $e(J^{[k_0+1]}, \mathbf{I}^{\mathbf{u}[\mathbf{k}]}; N) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \mathbf{u}^{\mathbf{k}}$ .
- (ii)  $e\left((J, \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R)_+); \mathfrak{R}\left(\mathbf{I}^{\mathbf{u}}; \frac{N}{0_N : I^\infty}\right)\right) = \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \mathbf{u}^{\mathbf{k}}$ .

Set  $\mathbb{S} = F_J(J, \mathbf{I}; R)$  and  $\mathbb{M} = F_J(J, \mathbf{I}; N)$ . Recall that

$$\tilde{e}(\mathbb{M}) = \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N).$$

Hence combining this fact with Note 2.3 and Corollary 2.5 yields:

**Corollary 2.6.**  $e\left(F_J(J, \mathbf{I}; \frac{N}{0_N : I^\infty})\right) = \tilde{e}(\mathbb{M}) = e\left((J, \mathfrak{R}(\mathbf{I}; R)_+); \mathfrak{R}(\mathbf{I}; \frac{N}{0_N : I^\infty})\right).$

**Remark 2.7.** If  $\text{ht} \frac{I + \text{Ann} N}{\text{Ann} N} > 0$ , then  $e(\mathbb{M})$  is the sum of all the mixed multiplicities of  $\mathbb{M}$  by [4, 20]. Hence  $e(\mathbb{M}) = \tilde{e}(\mathbb{M})$ . Thus  $e(F_J(J, \mathbf{I}; N)) = e\left(F_J(J, \mathbf{I}; \frac{N}{0_N : I^\infty})\right)$  and  $e((J, \mathfrak{R}(\mathbf{I}; R)_+); \mathfrak{R}(\mathbf{I}; N)) = e\left((J, \mathfrak{R}(\mathbf{I}; R)_+); \mathfrak{R}(\mathbf{I}; \frac{N}{0_N : I^\infty})\right)$  by Corollary 2.6.

### 3. Additivity and reduction formulas for mixed multiplicities

In this section, we prove additivity and reduction formulas for mixed multiplicities. And as an application of these formulas, we show that mixed multiplicities of arbitrary ideals are additive on exact sequences.

First, we have the following result for  $\mathbb{N}^d$ -graded  $S$ -modules.

**Theorem 3.1.** *Let  $S$  be a finitely generated standard  $\mathbb{N}^d$ -graded algebra over an artinian local ring  $A$  and let  $M$  be a finitely generated  $\mathbb{N}^d$ -graded  $S$ -module such that  $S_{(1,1,\dots,1)}$  is not contained in  $\sqrt{\text{Ann} M}$ . Denote by  $\Lambda$  the set of all homogeneous prime ideals  $P$  of  $S$  such that  $P \in \text{Supp}_{++} M$  and  $\dim \text{Proj}(S/P) = \dim \text{Supp}_{++} M$ . Then*

$$e(M; \mathbf{k}) = \sum_{P \in \Lambda} \ell(M_P) e(S/P; \mathbf{k}).$$

**Proof.** Denote by  $B_M(\mathbf{n})$  the Bahattacharya homogeneous polynomial of  $M$ . Remember that since  $S_{(1,1,\dots,1)} \not\subseteq \sqrt{\text{Ann} M}$ ,  $\deg B_M(\mathbf{n}) = \dim \text{Supp}_{++} M$  by [4, Theorem 4.1] (see [24, Remark 3.1]). Let

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_u = M$$

be a prime filtration of  $M$ , i.e.,  $M_{i+1}/M_i \cong S/P_i$  where  $P_i$  is a homogeneous prime ideal for all  $0 \leq i \leq u-1$ . Since  $S_{(1,1,\dots,1)} \not\subseteq \sqrt{\text{Ann} M}$ ,  $\emptyset \neq \Lambda \subseteq \text{Min}(S/\text{Ann} M)$  by [4, Lemma 1.1]. Consequently,  $\Lambda \subseteq \{P_0, P_1, \dots, P_{u-1}\}$ . Note that

$$\{P_0, P_1, \dots, P_{u-1}\} \subseteq \text{Supp} M.$$



Hence if  $P_i \notin \text{Supp}_{++} M$  then  $P_i \supseteq S_{++}$ . In this case,  $\left(\frac{S}{P_i}\right)_{\mathbf{n}} = 0$  for all  $\mathbf{n} \gg 0$  by [24, Proposition 2.7]. Therefore  $B_{S/P_i}(\mathbf{n}) = 0$ . If  $\dim \text{Proj}(S/P_i) < \dim \text{Supp}_{++} M$ , we have  $\deg B_{S/P_i}(\mathbf{n}) = \dim \text{Proj}(S/P_i) < \dim \text{Supp}_{++} M$  by [4, Theorem 4.1]. From the above facts, it follows that

$$\deg B_{S/P_i}(\mathbf{n}) < \dim \text{Supp}_{++} M$$

for all  $P_i \notin \Lambda$ . Hence  $B_M(\mathbf{n})$  is a sum of all the  $B_{S/P}(\mathbf{n})$  for  $P \in \Lambda$ , counted as many times as  $S/P$  appears as some  $\frac{M_{i+1}}{M_i}$ . This number is exactly the length of  $M_P$  because  $\Lambda \subseteq \text{Min}(S/\text{Ann} M)$ . Therefore  $B_M(\mathbf{n}) = \sum_{P \in \Lambda} \ell(M_P) B_{S/P}(\mathbf{n})$ . Set  $\dim \text{Supp}_{++} M = s$ . Remember that  $\mathbf{n}^{\mathbf{k}} := n_1^{k_1} \cdots n_d^{k_d}$ . Now since

$$B_{S/P}(\mathbf{n}) = \sum_{|\mathbf{k}|=s} e(S/P; \mathbf{k}) \frac{\mathbf{n}^{\mathbf{k}}}{k_1! \cdots k_d!} \text{ for any } P \in \Lambda,$$

$$B_M(\mathbf{n}) = \sum_{|\mathbf{k}|=s} \left[ \sum_{P \in \Lambda} \ell(M_P) e(S/P; \mathbf{k}) \right] \frac{\mathbf{n}^{\mathbf{k}}}{k_1! \cdots k_d!}.$$

Hence

$$\begin{aligned} & \sum_{|\mathbf{k}|=s} e(M; \mathbf{k}) \frac{\mathbf{n}^{\mathbf{k}}}{k_1! \cdots k_d!} \\ &= \sum_{|\mathbf{k}|=s} \left[ \sum_{P \in \Lambda} \ell(M_P) e(S/P; \mathbf{k}) \right] \frac{\mathbf{n}^{\mathbf{k}}}{k_1! \cdots k_d!}. \end{aligned}$$

Thus,

$$e(M; \mathbf{k}) = \sum_{P \in \Lambda} \ell(M_P) e(S/P; \mathbf{k}). \blacksquare$$

Although Theorem 3.1 is a general result for mixed multiplicities of multi-graded modules that is a general object for mixed multiplicities of ideals, up to now we can not give a proof for the case of mixed multiplicities of ideals in the following result by using this theorem.

**Theorem 3.2.** *Let  $(R, \mathfrak{n})$  be a noetherian local ring with maximal ideal  $\mathfrak{n}$ , infinite residue field  $k = R/\mathfrak{n}$ , ideals  $I_1, \dots, I_d$ , an  $\mathfrak{n}$ -primary  $J$ . Let  $N$  be a finitely generated  $R$ -module. Assume that  $I = I_1 \cdots I_d$  is not contained in  $\sqrt{\text{Ann} N}$ . Set  $\overline{N} = \frac{N}{0_N : I^\infty}$ . Denote by  $\Pi$  the set of all prime ideals  $\mathfrak{p}$  of  $R$  such that  $\mathfrak{p} \in \text{Min}(R/\text{Ann} \overline{N})$  and  $\dim R/\mathfrak{p} = \dim \overline{N}$ . Then we have*

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = \sum_{\mathfrak{p} \in \Pi} \ell(N_{\mathfrak{p}}) e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}).$$

**Remark 3.3.** Recall that  $\Pi$  is the set of prime ideals  $\mathfrak{p}$  such that  $\mathfrak{p} \in \text{Min}(R/\text{Ann}\overline{N})$  and  $\dim R/\mathfrak{p} = \dim \overline{N}$ . It is easy seen that

$$\Pi = \left\{ \mathfrak{p} \in \text{Ass}\left(\frac{R}{\text{Ann}\overline{N}}\right) \mid \dim R/\mathfrak{p} = \dim \overline{N} \right\}.$$

Since  $\text{Ann}\overline{N} = \text{Ann}N : I^\infty$ ,  $\frac{R}{\text{Ann}\overline{N}} = \frac{R}{\text{Ann}N : I^\infty}$ . Consequently

$$\begin{aligned} \Pi &= \left\{ \mathfrak{p} \in \text{Ass}\left(\frac{R}{\text{Ann}N : I^\infty}\right) \mid \dim R/\mathfrak{p} = \dim \overline{N} \right\} \\ &= \left\{ \mathfrak{p} \in \text{Ass}\left(\frac{R}{\text{Ann}N}\right) \mid \mathfrak{p} \not\supseteq I \text{ and } \dim R/\mathfrak{p} = \dim \overline{N} \right\}. \end{aligned}$$

If  $\mathfrak{p} \in \Pi$ ,  $\overline{N}_{\mathfrak{p}} = N_{\mathfrak{p}}$  because  $I \not\subseteq \mathfrak{p}$ . Since  $\ell(N_{\mathfrak{p}}) = \ell(\overline{N}_{\mathfrak{p}}) < +\infty$ ,  $\mathfrak{p} \in \text{Min}(\frac{R}{\text{Ann}N})$ . Hence  $\Pi = \left\{ \mathfrak{p} \in \text{Min}\left(\frac{R}{\text{Ann}N}\right) \mid \mathfrak{p} \not\supseteq I \text{ and } \dim R/\mathfrak{p} = \dim \overline{N} \right\}$ . In the case that  $\text{ht}\frac{I + \text{Ann}N}{\text{Ann}N} > 0$ ,  $\dim \overline{N} = \dim N$  and  $\mathfrak{p} \not\supseteq I$  for any  $\mathfrak{p} \in \text{Min}(\frac{R}{\text{Ann}N})$ . Consequently  $\Pi = \left\{ \mathfrak{p} \in \text{Min}\left(\frac{R}{\text{Ann}N}\right) \mid \dim R/\mathfrak{p} = \dim N \right\}$ .

Our approach is based on multiplicity formulas of multi-graded Rees modules with respect to powers of ideals that gave in Proposition 2.4 via linking homogeneous prime ideals which are in  $\text{Min}(\mathfrak{R}(\mathbf{I}; R)/\text{Ann} \mathfrak{R}(\mathbf{I}; N))$  of maximal coheight and prime ideals in  $\Pi$  by the following lemma.

**Lemma 3.4.** *Let  $N$  be a finitely generated  $R$ -module and let  $I_1, \dots, I_d$  be ideals of  $R$  such that  $\text{ht}\frac{I + \text{Ann}N}{\text{Ann}N} > 0$  ( $I = I_1 \cdots I_d$ ). Denote by  $\Lambda$  the set of homogeneous prime ideals  $P$  of the Rees algebras  $\mathfrak{R}(\mathbf{I}; R)$  such that  $P \in \text{Min}(\mathfrak{R}(\mathbf{I}; R)/\text{Ann}\mathfrak{R}(\mathbf{I}; N))$  and  $\dim \mathfrak{R}(\mathbf{I}; R)/P = \dim \mathfrak{R}(\mathbf{I}; N)$ , and denote by  $\Pi$  the set of prime ideals of  $R$  such that  $\mathfrak{p} \in \text{Min}(R/\text{Ann}N)$  and  $\dim R/\mathfrak{p} = \dim N$ . Then there is an one-to-one correspondence between the set of prime ideals  $\Pi$  and the set of prime ideals  $\Lambda$  given by*

$$\mathfrak{p} \mapsto P = \bigoplus_{n_1, \dots, n_d \geq 0} (\mathfrak{p} \cap I_1^{n_1} \cdots I_d^{n_d}).$$

**Proof.** First, remember that since  $\text{ht}\frac{I + \text{Ann}N}{\text{Ann}N} > 0$ ,  $\dim \mathfrak{R}(\mathbf{I}; N) = \dim N + d$ . Note that  $\Lambda \subseteq \text{Ass}_{\mathfrak{R}(\mathbf{I}; R)} \mathfrak{R}(\mathbf{I}; N)$  and  $\Pi \subseteq \text{Ass}_R N$  and

$$\text{Ann} \mathfrak{R}(\mathbf{I}; N) = \bigoplus_{n_1, \dots, n_d \geq 0} (\text{Ann}N \cap I_1^{n_1} \cdots I_d^{n_d}).$$

Now, if  $\mathfrak{p}$  is an ideal in  $\Pi$ , then it can be verified that

$$P = \bigoplus_{n_1, \dots, n_d \geq 0} (\mathfrak{p} \cap I_1^{n_1} \cdots I_d^{n_d})$$

is a homogeneous prime ideal of  $\mathfrak{R}(\mathbf{I}; R)$  and  $\text{Ann } \mathfrak{R}(\mathbf{I}; N) \subseteq P$ .

**Note 3.5.** If  $\mathfrak{q}$  is a prime ideal of  $R$  and  $I \not\subseteq \mathfrak{q}$  then  $\frac{I + \mathfrak{q}}{\mathfrak{q}} \neq 0$ . Since  $R/\mathfrak{q}$  is an integral domain and  $\frac{I + \mathfrak{q}}{\mathfrak{q}} \neq 0$ ,  $\text{ht} \frac{I + \mathfrak{q}}{\mathfrak{q}} > 0$ . Therefore for any  $\mathfrak{p} \in \Pi$ ,  $\text{ht} \frac{I + \mathfrak{p}}{\mathfrak{p}} > 0$  because  $I \not\subseteq \mathfrak{p}$  by Remark 3.3.

It is easily seen that

$$\begin{aligned} \mathfrak{R}(\mathbf{I}; R)/P &= \bigoplus_{n_1, \dots, n_d \geq 0} \frac{I_1^{n_1} \cdots I_d^{n_d}}{\mathfrak{p} \cap I_1^{n_1} \cdots I_d^{n_d}} \\ &\cong \bigoplus_{n_1, \dots, n_d \geq 0} \frac{I_1^{n_1} \cdots I_d^{n_d} + \mathfrak{p}}{\mathfrak{p}} = \mathfrak{R}(\mathbf{I}; R/\mathfrak{p}). \end{aligned}$$

Since  $\text{ht} \frac{I + \mathfrak{p}}{\mathfrak{p}} > 0$  by Note 3.5,  $\dim \mathfrak{R}(\mathbf{I}; R/\mathfrak{p}) = \dim R/\mathfrak{p} + d$ . Since  $\mathfrak{p} \in \Pi$ ,  $\dim R/\mathfrak{p} = \dim N$ . Hence  $\dim \mathfrak{R}(\mathbf{I}; R)/P = \dim \mathfrak{R}(\mathbf{I}; N)$ . So  $P \in \Lambda$ .

Next, suppose that  $P$  is an ideal in  $\Lambda$ . Then  $P$  is an associated prime ideal of  $\mathfrak{R}(\mathbf{I}; N)$ . Hence  $P$  is homogeneous and there is a homogeneous element  $x \in \mathfrak{R}(\mathbf{I}; N)$  such that  $P = 0 : x$ . Set  $\mathfrak{p} = P \cap R$ . Then  $\mathfrak{p} = \{a \in R \mid ax = 0\}$  and  $\text{Ann} N \subseteq \mathfrak{p}$ . Set  $P = \bigoplus_{n_1, \dots, n_d \geq 0} P_{(n_1, \dots, n_d)}$ , we have

$$P_{(n_1, \dots, n_d)} = \{a \in I_1^{n_1} \cdots I_d^{n_d} \mid ax = 0\}.$$

It implies that  $P_{(n_1, \dots, n_d)} = \mathfrak{p} \cap I_1^{n_1} \cdots I_d^{n_d}$ . Therefore  $P$  has the form

$$P = \bigoplus_{n_1, \dots, n_d \geq 0} (\mathfrak{p} \cap I_1^{n_1} \cdots I_d^{n_d}).$$

Consequently  $\mathfrak{R}(\mathbf{I}; R)/P \cong \mathfrak{R}(\mathbf{I}; R/\mathfrak{p})$ . Since  $P \in \Lambda$  and  $\text{ht} \frac{I + \text{Ann} N}{\text{Ann} N} > 0$ ,

$$\dim \mathfrak{R}(\mathbf{I}; R)/P = \dim \mathfrak{R}(\mathbf{I}; N) = \dim N + d.$$

Note that

$$\dim \mathfrak{R}(\mathbf{I}; R/\mathfrak{p}) \leq \dim R/\mathfrak{p} + d.$$

Consequently  $\dim R/\mathfrak{p} \geq \dim N$ . Hence since  $\text{Ann} N \subseteq \mathfrak{p}$ ,  $\dim R/\mathfrak{p} = \dim N$ . Thus,  $\mathfrak{p} \in \Pi$ . The above facts follow that there is a bijection between the set  $\Pi$  and the set  $\Lambda$  given by  $\mathfrak{p} \mapsto P = \bigoplus_{n_1, \dots, n_d \geq 0} (\mathfrak{p} \cap I_1^{n_1} \cdots I_d^{n_d})$ . ■

**The proof of Theorem 3.2:** Let  $u_1, \dots, u_d$  be positive integers. Remember that  $\mathbf{I}^{\mathbf{u}} = I_1^{u_1}, \dots, I_d^{u_d}$ . Set  $\overline{N} = \frac{N}{0_N : I^\infty}$  and  $q = \dim \overline{N}$ . Denote by  $\Lambda_{\mathbf{u}}$  the set of homogeneous prime ideals  $P$  of the Rees algebra  $\mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R) = \mathfrak{R}(I_1^{u_1}, \dots, I_d^{u_d}; R)$  such that  $P \in \text{Min}(\mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R)/\text{Ann}\mathfrak{R}(\mathbf{I}^{\mathbf{u}}; \overline{N}))$  and  $\dim \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R)/P = \dim \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; \overline{N})$ . Recall that

$$\Pi = \left\{ \mathfrak{p} \in \text{Min}\left(\frac{R}{\text{Ann}\overline{N}}\right) \mid \dim R/\mathfrak{p} = \dim \overline{N} \right\}.$$

By [5, Theorem 11.2.4], we have

$$\begin{aligned} & e((J, \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R)_+); \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; \overline{N})) \\ &= \sum_{P \in \Lambda_{\mathbf{u}}} \ell(\mathfrak{R}(\mathbf{I}^{\mathbf{u}}; \overline{N})_P) e((J, \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R)_+); \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R)/P). \end{aligned} \quad (3)$$

Remember that  $\text{ht} \frac{I + \text{Ann}\overline{N}}{\text{Ann}\overline{N}} > 0$ . In this case, if  $P \in \Lambda_{\mathbf{u}}$  and  $\mathfrak{p} = P \cap R$ , we have

$$P = \bigoplus_{n_1, \dots, n_d \geq 0} (\mathfrak{p} \cap (I_1^{u_1})^{n_1} \cdots (I_d^{u_d})^{n_d})$$

and  $\mathfrak{p} \in \Pi$  by Lemma 3.4. Since  $\mathfrak{p} \not\supseteq I$  by Remark 3.3, it follows that

$$\mathfrak{R}(\mathbf{I}^{\mathbf{u}}; \overline{N})_P = \overline{N}_{\mathfrak{p}}.$$

Hence since

$$\mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R)/P \cong \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R/\mathfrak{p})$$

and by (3) we obtain

$$e((J, \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R)_+); \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; \overline{N})) = \sum_{\mathfrak{p} \in \Pi} \ell(\overline{N}_{\mathfrak{p}}) e((J, \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R/\mathfrak{p})_+); \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R/\mathfrak{p})). \quad (4)$$

Recall that  $\mathfrak{p} \in \Pi$ ,  $\text{ht} \frac{I + \mathfrak{p}}{\mathfrak{p}} > 0$  by Note 3.5. Hence by Corollary 2.5(ii) and Proposition 2.4, we respectively get

$$e((J, \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R)_+); \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; \overline{N})) = \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \mathbf{u}^{\mathbf{k}} \quad (5)$$

and

$$e((J, \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R/\mathfrak{p})_+); \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; R/\mathfrak{p})) = \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}) \mathbf{u}^{\mathbf{k}}. \quad (6)$$

From (4), (5) and (6), it follows that

$$\begin{aligned}
& \sum_{k_0+|\mathbf{k}|=q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \mathbf{u}^{\mathbf{k}} \\
&= \sum_{\mathfrak{p} \in \Pi} \ell(\overline{N}_{\mathfrak{p}}) \left( \sum_{k_0+|\mathbf{k}|=q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}) \mathbf{u}^{\mathbf{k}} \right) \\
&= \sum_{k_0+|\mathbf{k}|=q-1} \left( \sum_{\mathfrak{p} \in \Pi} \ell(\overline{N}_{\mathfrak{p}}) e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}) \right) \mathbf{u}^{\mathbf{k}}.
\end{aligned}$$

Therefore

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = \sum_{\mathfrak{p} \in \Pi} \ell(\overline{N}_{\mathfrak{p}}) e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}).$$

Recall that  $\ell(\overline{N}_{\mathfrak{p}}) = \ell(N_{\mathfrak{p}})$  by Remark 3.3. Thus

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = \sum_{\mathfrak{p} \in \Pi} \ell(N_{\mathfrak{p}}) e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}). \blacksquare$$

Note that if  $\text{ht} \frac{I + \text{Ann} N}{\text{Ann} N} > 0$  then  $\Pi = \{\mathfrak{p} \in \text{Min} \left( \frac{R}{\text{Ann} N} \right) \mid \dim R/\mathfrak{p} = \dim N\}$  by Remark 3.3. Hence by Theorem 3.2, we obtain the following result.

**Corollary 3.6.** *Let  $(R, \mathfrak{n})$  be a noetherian local ring with maximal ideal  $\mathfrak{n}$  and infinite residue field  $k = R/\mathfrak{n}$ , ideals  $I_1, \dots, I_d$  and an  $\mathfrak{n}$ -primary ideal  $J$ . Let  $N$  be a finitely generated  $R$ -module. Set  $I = I_1 \cdots I_d$ . Assume that  $\text{ht} \frac{I + \text{Ann} N}{\text{Ann} N} > 0$ . Denote by  $\Pi$  the set of all prime ideals  $\mathfrak{p}$  of  $R$  such that  $\mathfrak{p} \in \text{Min}(R/\text{Ann} N)$  and  $\dim R/\mathfrak{p} = \dim N$ . Then we have*

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = \sum_{\mathfrak{p} \in \Pi} \ell(N_{\mathfrak{p}}) e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}).$$

Let  $I_1, \dots, I_d$  be  $\mathfrak{n}$ -primary ideals of  $R$ . Set  $\dim N = q$ . Denote by  $P(n_1, \dots, n_d)$  the Hilbert-Samuel polynomial of the Hilbert-Samuel function  $\ell_A \left( \frac{N}{I_1^{n_1} \cdots I_d^{n_d} N} \right)$ . For any  $1 \leq i \leq d$ , denote by  $Q_i(n_1, \dots, n_d)$  the Hilbert-Samuel polynomial of the Hilbert-Samuel function  $\ell_A \left( \frac{I_1^{n_1} \cdots I_i^{n_i} \cdots I_d^{n_d} N}{I_1^{n_1} \cdots I_i^{n_i+1} \cdots I_d^{n_d} N} \right)$ . Then we have  $\deg P(n_1, \dots, n_d) = q$  and

$$P(n_1, \dots, n_i + 1, \dots, n_d) - P(n_1, \dots, n_i, \dots, n_d) = Q_i(n_1, \dots, n_i, \dots, n_d).$$

Write the terms of total degree  $q$  in  $P(\mathbf{n})$  in the form  $\sum_{|\mathbf{k}|=q} e(\mathbf{I}^{[\mathbf{k}]}; N) \frac{\mathbf{n}^{\mathbf{k}}}{k_1! \cdots k_d!}$ . Since  $k_1 + \cdots + k_d = |\mathbf{k}| = q > 0$ , there exists  $1 \leq j \leq d$  such that  $k_j > 0$ . It is

easy to check that  $\frac{e(\mathbf{I}^{[\mathbf{k}]}; N)}{k_1! \cdots (k_j - 1)! \cdots k_d!} n_1^{k_1} \cdots n_j^{k_j - 1} \cdots n_d^{k_d}$  is a term of total degree  $q - 1$  in  $Q_j(\mathbf{n})$ . So  $e(\mathbf{I}^{[\mathbf{k}]}; N)$  as in [5] is exactly the mixed multiplicity of  $N$  with respect to  $(I_1, \dots, I_j, \dots, I_d)$  of the type  $(k_1, \dots, k_j, \dots, k_d)$  defined in Section 2 with  $I_j$  playing the role of  $J$ . Therefore, for any non-negative integers  $k_1, \dots, k_d$  with  $k_1 + \cdots + k_d = |\mathbf{k}| = q$ , one also calls  $e(\mathbf{I}^{[\mathbf{k}]}; N)$  the mixed multiplicity of  $N$  with respect to  $(I_1, \dots, I_d)$  of the type  $(k_1, \dots, k_d)$ .

Then as a consequence of Corollary 3.6, we get the following result.

**Corollary 3.7** [5, Theorem 17.4.8]. *Let  $(R, \mathfrak{n})$  be a noetherian local ring with maximal ideal  $\mathfrak{n}$  and infinite residue field  $k = R/\mathfrak{n}$ , and  $\mathfrak{n}$ -primary ideals  $I_1, \dots, I_d$ . Let  $N$  be a finitely generated  $R$ -module of Krull dimension  $\dim N > 0$ . Denote by  $\Pi$  the set of all prime ideals  $\mathfrak{p}$  of  $R$  such that  $\mathfrak{p} \in \text{Min}(R/\text{Ann}N)$  and  $\dim R/\mathfrak{p} = \dim N$ . Assume that  $k_1, \dots, k_d$  are non-negative integers with  $k_1 + \cdots + k_d = \dim N$ . Then we have*

$$e(\mathbf{I}^{[\mathbf{k}]}; N) = \sum_{\mathfrak{p} \in \Pi} \ell(N_{\mathfrak{p}}) e(\mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}).$$

**Proof.** Since  $\dim N > 0$  and  $I = I_1 \cdots I_d$  is an  $\mathfrak{n}$ -primary ideal,  $\text{ht} \frac{I + \text{Ann}N}{\text{Ann}N} > 0$ . Hence the proof is immediate from Corollary 3.6. ■

**Remark 3.8.** Let  $W_1, W_2, W_3$  be finitely generated  $R$ -modules and let  $I_1, \dots, I_d$  be ideals of  $R$  such that  $I = I_1 \cdots I_d \not\subseteq \sqrt{\text{Ann}W_i}$  for all  $i = 1, 2, 3$ . Let

$$0 \longrightarrow W_1 \longrightarrow W_3 \longrightarrow W_2 \longrightarrow 0$$

be a short exact sequence of  $R$ -modules. For any  $i = 1, 2, 3$ , set  $\overline{W}_i = \frac{W_i}{0_{W_i} : I^\infty}$  and  $p_i = \dim \overline{W}_i$ . Denote by  $\Pi_i$  the set of prime ideals such that  $\mathfrak{p} \in \text{Min}(R/\text{Ann}\overline{W}_i)$  and  $\dim R/\mathfrak{p} = p_i$ . Set  $\Omega = \Pi_1 \cup \Pi_2 \cup \Pi_3$ . For any  $\mathfrak{p} \in \Omega$ , we have always short exact sequences

$$0 \longrightarrow (W_1)_{\mathfrak{p}} \longrightarrow (W_3)_{\mathfrak{p}} \longrightarrow (W_2)_{\mathfrak{p}} \longrightarrow 0.$$

If  $p_j < p_i$  and  $p_k = \{p_1, p_2, p_3\} \setminus \{p_i, p_j\}$  then for any  $\mathfrak{p} \in \Pi_i$ , we get  $\dim \overline{W}_j < \dim R/\mathfrak{p}$ , and hence  $\mathfrak{p} \not\subseteq \text{Ann}\overline{W}_j$ . In this case,  $(\overline{W}_j)_{\mathfrak{p}} = 0$ . By Remark 3.3,  $(W_j)_{\mathfrak{p}} = (\overline{W}_j)_{\mathfrak{p}}$ . Hence  $(W_j)_{\mathfrak{p}} = 0$ . Thus  $0 \neq (W_i)_{\mathfrak{p}} = (W_k)_{\mathfrak{p}}$ . This argument proves that if  $p_j < p_i$  then  $p_i = p_k$  and  $\Pi_i = \Pi_k$ , moreover,  $p_3 = \max\{p_1, p_2\}$ .

Using Theorem 3.2, now we prove that the mixed multiplicities of arbitrary ideals are additive on short exact sequences by the following result.

**Corollary 3.9.** *Keep the notations as in Remark 3.8. Let  $J$  be an  $\mathfrak{n}$ -primary ideal. Set  $\mathfrak{J} = (J, \mathfrak{R}(\mathbf{I}; R)_+)$ . Assume that  $\dim \overline{W}_3 = k_0 + k_1 + \cdots + k_d + 1$ . Then the following statements hold.*

(i) *If  $\dim \overline{W}_1 = \dim \overline{W}_2 = \dim \overline{W}_3$  then*

$$\begin{aligned} (a) : e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_3) &= e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_1) + e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_2); \\ (b) : e(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \overline{W}_3)) &= e(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \overline{W}_1)) + e(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \overline{W}_2)). \end{aligned}$$

(ii) *If  $h \neq k = 1, 2$  and  $\dim \overline{W}_3 > \dim \overline{W}_h$  then*

$$\begin{aligned} (a) : e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_3) &= e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_k); \\ (b) : e(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \overline{W}_3)) &= e(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \overline{W}_k)). \end{aligned}$$

**Proof.** The proof of (i): Since  $p_1 = p_2 = p_3$ , by Theorem 3.2 we have

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_i) = \sum_{\mathfrak{p} \in \Pi_i} \ell(W_i)_{\mathfrak{p}} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p})$$

for  $i = 1, 2, 3$ . Let  $\mathfrak{p} \in \Omega \setminus \Pi_i$ . Since  $\dim R/\mathfrak{p} = p_i$ ,  $\mathfrak{p} \not\subseteq \text{Ann} \overline{W}_i$ . Consequently,  $(\overline{W}_i)_{\mathfrak{p}} = 0$ . By Remark 3.3,  $(W_i)_{\mathfrak{p}} = (\overline{W}_i)_{\mathfrak{p}}$ . So  $(W_i)_{\mathfrak{p}} = 0$ . From this it follows that

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_i) = \sum_{\mathfrak{p} \in \Pi_i} \ell(W_i)_{\mathfrak{p}} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}) = \sum_{\mathfrak{p} \in \Omega} \ell(W_i)_{\mathfrak{p}} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p})$$

for all  $i = 1, 2, 3$ . Therefore by Theorem 3.2, we obtain

$$\begin{aligned} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_3) &= \sum_{\mathfrak{p} \in \Omega} \ell(W_3)_{\mathfrak{p}} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}) \\ &= \sum_{\mathfrak{p} \in \Omega} (\ell(W_1)_{\mathfrak{p}} + \ell(W_2)_{\mathfrak{p}}) e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}) \\ &= \sum_{\mathfrak{p} \in \Omega} \ell(W_1)_{\mathfrak{p}} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}) + \sum_{\mathfrak{p} \in \Omega} \ell(W_2)_{\mathfrak{p}} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}) \\ &= e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_1) + e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_2). \end{aligned}$$

Hence we get (a) of (i). By Corollary 2.5(ii) we have (b) of (i). The case that  $p_3 > p_h$ : By Remark 3.8,  $p_3 = p_k$ ,  $\Pi_3 = \Pi_k$  and  $(W_3)_{\mathfrak{p}} = (W_k)_{\mathfrak{p}}$  for all  $\mathfrak{p} \in \Pi_3 = \Pi_k$ . Consequently, we obtain (ii) by Corollary 2.5(ii) and since

$$\begin{aligned} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_3) &= \sum_{\mathfrak{p} \in \Pi_3} \ell(W_3)_{\mathfrak{p}} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}) \\ &= \sum_{\mathfrak{p} \in \Pi_k} \ell(W_k)_{\mathfrak{p}} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; R/\mathfrak{p}) \\ &= e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_k). \blacksquare \end{aligned}$$

**Remark 3.10.** Now, if we assign the mixed multiplicities of modules  $W_i$  :

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_i) = 0$$

to the case that  $k_0 + \dots + k_d > \dim \overline{W}_i - 1$ , then from Corollary 3.9, we immediately get that: if  $k_0 + |\mathbf{k}| = \dim \overline{W}_3 - 1$  then

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_3) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_1) + e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; W_2).$$

It is natural to suppose that the proof of Theorem 3.2 will be based on Corollary 3.9. Hence one of obstructions in proving Theorem 3.2 is Corollary 3.9. This is a motivation to help us giving the proof of Theorem 3.2 as in this paper.

## 4. Filter-regular sequences of multi-graded modules

In this section, we explore the relationship between filter-regular sequences of the multi-graded  $F_J(J, \mathbf{I}; R)$ -module  $F_J(J, \mathbf{I}; N)$  and weak-(FC)-sequences of ideals that will be used in the proofs of Section 5.

The concept of filter-regular sequences was introduced by Stuckrad and Vogel in [15]. The theory of filter-regular sequences became an important tool to study some classes of singular rings and has been continually developed (see e.g. [2, 6, 18, 19, 24]).

**Definition 4.1.** Let  $S = \bigoplus_{n_1, \dots, n_d \geq 0} S_{(n_1, \dots, n_d)}$  be a finitely generated standard  $\mathbb{N}^d$ -graded algebra over an artinian local ring  $A$  and let  $M = \bigoplus_{n_1, \dots, n_d \geq 0} M_{(n_1, \dots, n_d)}$  be a finitely generated  $\mathbb{N}^d$ -graded  $S$ -module. Let  $S_{(1,1,\dots,1)}$  be not contained in  $\sqrt{\text{Ann} M}$ . Then a homogeneous element  $x \in S$  is called an  $S_{++}$ -filter-regular element with respect to  $M$  if  $(0_M : x)_{(n_1, \dots, n_d)} = 0$  for all large  $n_1, \dots, n_d$ . Let  $x_1, \dots, x_t$  be homogeneous elements in  $S$ . We call that  $x_1, \dots, x_t$  is an  $S_{++}$ -filter-regular sequence with respect to  $M$  if  $x_i$  is an  $S_{++}$ -filter-regular element with respect to  $\frac{M}{(x_1, \dots, x_{i-1})M}$  for all  $i = 1, \dots, t$ .

**Remark 4.2.** If  $S_{(1,1,\dots,1)} \subseteq \sqrt{\text{Ann} M}$  then  $(0_M : x)_{(n_1, \dots, n_d)} \subseteq M_{(n_1, \dots, n_d)} = 0$  for all large  $n_1, \dots, n_d$ . Hence any homogeneous element of  $S$  always has the property of an  $S_{++}$ -filter-regular element. This fact only obstruct and do not carry usefulness. That is why in Definition 4.1, one has to exclude the case that  $S_{(1,1,\dots,1)} \subseteq \sqrt{\text{Ann} M}$  in defining  $S_{++}$ -filter-regular elements.



**Note 2.3.** If  $S_{(1,1,\dots,1)} \not\subseteq \sqrt{\text{Ann}M}$ , then by [24], a homogeneous element  $x \in S$  is an  $S_{++}$ -filter-regular element with respect to  $M$  if and only if  $x \notin P$  for any  $P \in \text{Ass}_S M$  and  $P$  does not contain  $S_{++}$ . That means  $x \notin \bigcup_{S_{++} \not\subseteq P, P \in \text{Ass}_S M} P$ . In this case, for any  $1 \leq i \leq d$ , there exists an  $S_{++}$ -filter-regular element  $x \in S_i \setminus \mathfrak{m}S_i$ .

Remember that the positivity and the relationship between mixed multiplicities and Hilbert-Samuel multiplicities of ideals have attracted much attention (see e.g. [7, 8, 9, 11, 14, 16, 19, 21, 22, 23, 25]). In past years, using different sequences, one expressed mixed multiplicities into Hilbert-Samuel multiplicity, for instance: Risler-Teissier in 1973 [17] by superficial sequences and Rees in 1984 [13] by joint reductions; Viet in 2000 [21] by (FC)-sequences (see e.g. [3, 11, 25]).

**Definition 4.4** [21]. Let  $(R, \mathfrak{n})$  be a noetherian local ring with maximal ideal  $\mathfrak{n}$ , infinite residue field  $k = R/\mathfrak{n}$  and let  $N$  be a finitely generated  $R$ -module. Let  $I_1, \dots, I_d$  be ideals such that  $I_1 \cdots I_d$  is not contained in  $\sqrt{\text{Ann}N}$ . Set  $I = I_1 \cdots I_d$ . An element  $x \in R$  is called an *(FC)-element of  $N$  with respect to  $(I_1, \dots, I_d)$*  if there exists  $i \in \{1, \dots, d\}$  such that  $x \in I_i$  and the following conditions are satisfied:

- (i)  $x$  is an  $I$ -filter-regular element with respect to  $N$ , i.e.,  $0_N : x \subseteq 0_N : I^\infty$ .
- (ii)  $xN \cap I_1^{n_1} \cdots I_i^{n_i+1} \cdots I_d^{n_d} N = xI_1^{n_1} \cdots I_i^{n_i} \cdots I_d^{n_d} N$  for all  $n_1, \dots, n_d \gg 0$ .
- (iii)  $\dim N/(xN : I^\infty) = \dim N/0_N : I^\infty - 1$ .

We call  $x$  a *weak-(FC)-element of  $N$  with respect to  $(I_1, \dots, I_d)$*  if  $x$  satisfies the conditions (i) and (ii).

Let  $x_1, \dots, x_t$  be a sequence in  $R$ . For any  $0 \leq i < t$ , set  $N_i = \frac{N}{(x_1, \dots, x_i)N}$ . Then  $x_1, \dots, x_t$  is called a *weak-(FC)-sequence of  $N$  with respect to  $(I_1, \dots, I_d)$*  if  $x_{i+1}$  is a weak-(FC)-element of  $N_i$  with respect to  $(I_1, \dots, I_d)$  for all  $i = 0, \dots, t-1$ .

$x_1, \dots, x_t$  is called an *(FC)-sequence of  $N$  with respect to  $(I_1, \dots, I_d)$*  if  $x_{i+1}$  is an (FC)-element of  $N_i$  with respect to  $(I_1, \dots, I_d)$  for all  $i = 0, \dots, t-1$ .

Recall that

$$\begin{aligned} \tilde{e}(M) &= \sum_{|\mathbf{k}|=\ell-1} e(M; \mathbf{k}); \\ S_i &= S_{(0, \dots, \underbrace{1}_i, \dots, 0)} \text{ for all } i = 1, \dots, d; \\ \mathbb{S} &= F_J(J, \mathbf{I}; R) = \bigoplus_{n_0, n_1, \dots, n_d \geq 0} \frac{J^{n_0} I_1^{n_1} \cdots I_d^{n_d}}{J^{n_0+1} I_1^{n_1} \cdots I_d^{n_d}}; \\ \mathbb{M} &= F_J(J, \mathbf{I}; N) = \bigoplus_{n_0, n_1, \dots, n_d \geq 0} \frac{J^{n_0} I_1^{n_1} \cdots I_d^{n_d} N}{J^{n_0+1} I_1^{n_1} \cdots I_d^{n_d} N}. \end{aligned}$$

Set  $S_i = \bigoplus_{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d \geq 0; n_i=0} S_{(n_1, \dots, n_d)}$  and  $M_i = S_i M_{(0, \dots, 0)}$ .

**Proposition 4.5.** *Let  $x \in I_i$  be a weak-(FC)-element with respect to  $(J, I_1, \dots, I_d)$  of  $N$ . Denote by  $\bar{x}$  and  $\bar{I}_i$  the images of  $x$  and  $I_i$  in  $\mathbb{S}_i$ , respectively. Then we have:*

- (i)  $\bar{x}$  is an  $\mathbb{S}_{++}$ -filter-regular element with respect to  $\mathbb{M}$ .
- (ii)  $\dim(\mathbb{M}/\bar{x}\mathbb{M})^\Delta = \dim \frac{N}{xN : I^\infty}$  and  $\tilde{e}(\mathbb{M}/\bar{x}\mathbb{M}) = \tilde{e}(F_J(J, \mathbf{I}; \frac{N}{xN}))$ .
- (iii)  $\mathbb{S}/\bar{I}_i\mathbb{S} \cong F_J(J, I_1, \dots, I_{i-1}, I_{i+1}, \dots, I_d; R) \cong \mathbb{S}_{\hat{i}}$  and  $\mathbb{M}/\bar{I}_i\mathbb{M} \cong F_J(J, I_1, \dots, I_{i-1}, I_{i+1}, \dots, I_d; N) \cong \mathbb{M}_{\hat{i}}$ .

**Proof.** We have  $(0_N : I^\infty) \cap J^m I_1^{m_1} \dots I_d^{m_d} N = 0$  for all  $m, m_1, \dots, m_d \gg 0$  by Artin-Rees lemma. Since  $x$  is an  $I$ -filter-regular element with respect to  $M$ ,

$$(0_N : x) \bigcap J^m I_1^{m_1} \dots I_d^{m_d} N \subseteq (0_N : I^\infty) \bigcap J^m I_1^{m_1} \dots I_d^{m_d} N = 0$$

for all  $m, m_1, \dots, m_d \gg 0$ . From this it follows that

$$\begin{aligned} & (J^{n+1} I_1^{n_1} \dots I_i^{n_i+1} \dots I_d^{n_d} N : x) \bigcap J^n I_1^{n_1} \dots I_i^{n_i} \dots I_d^{n_d} N \\ &= \left[ (xN \bigcap J^{n+1} I_1^{n_1} \dots I_i^{n_i+1} \dots I_d^{n_d} N) : x \right] \bigcap J^n I_1^{n_1} \dots I_i^{n_i} \dots I_d^{n_d} N \\ &= \left[ x J^{n+1} I_1^{n_1} \dots I_i^{n_i} \dots I_d^{n_d} N : x \right] \bigcap J^n I_1^{n_1} \dots I_i^{n_i} \dots I_d^{n_d} N \\ &= \left[ J^{n+1} I_1^{n_1} \dots I_i^{n_i} \dots I_d^{n_d} N + 0_N : x \right] \bigcap J^n I_1^{n_1} \dots I_i^{n_i} \dots I_d^{n_d} N \\ &= J^{n+1} I_1^{n_1} \dots I_i^{n_i} \dots I_d^{n_d} N + (0_N : x) \bigcap J^n I_1^{n_1} \dots I_i^{n_i} \dots I_d^{n_d} N \\ &= J^{n+1} I_1^{n_1} \dots I_i^{n_i} \dots I_d^{n_d} N \end{aligned}$$

for all  $n, n_1, \dots, n_d \gg 0$ . Hence  $[0_{\mathbb{M}} : \bar{x}]_{(n, n_1, \dots, n_d)} = 0$  for all  $n, n_1, \dots, n_d \gg 0$ . Thus,

$\bar{x}$  is an  $\mathbb{S}_{++}$ -filter-regular element. We get (i). It can be verified that

$$\begin{aligned} [\mathbb{M}/\bar{x}\mathbb{M}]_{(m, m_1, \dots, m_d)} &\cong \frac{J^m I_1^{m_1} \dots I_d^{m_d} N}{J^{m+1} I_1^{m_1} \dots I_d^{m_d} N + x J^m I_1^{m_1} \dots I_i^{m_i-1} \dots I_d^{m_d} N} \quad \text{and} \\ \left[ F_J(J, \mathbf{I}; \frac{N}{xN}) \right]_{(m, m_1, \dots, m_d)} &= \left[ \bigoplus_{n, n_1, \dots, n_d \geq 0} \frac{J^n I_1^{n_1} \dots I_d^{n_d} (N/xN)}{J^{n+1} I_1^{n_1} \dots I_d^{n_d} (N/xN)} \right]_{(m, m_1, \dots, m_d)} \\ &\cong \frac{J^m I_1^{m_1} \dots I_d^{m_d} N + xN}{J^{m+1} I_1^{m_1} \dots I_d^{m_d} N + xN} \cong \frac{J^m I_1^{m_1} \dots I_d^{m_d} N}{J^{m+1} I_1^{m_1} \dots I_d^{m_d} N + xN \bigcap J^m I_1^{m_1} \dots I_d^{m_d} N}. \end{aligned}$$

Since  $x$  is a weak-(FC)-element,

$$xN \bigcap J^m I_1^{m_1} \cdots I_d^{m_d} N = xJ^m I_1^{m_1} \cdots I_i^{m_i-1} \cdots I_d^{m_d} N$$

for all  $m, m_1, \dots, m_d \gg 0$ . Hence

$$[\mathbb{M}/\bar{x}\mathbb{M}]_{(m, m_1, \dots, m_d)} \cong \left[ F_J(J, \mathbf{I}; \frac{N}{xN}) \right]_{(m, m_1, \dots, m_d)}$$

for all  $m, m_1, \dots, m_d \gg 0$ . From this it follows that

$$\dim (\mathbb{M}/\bar{x}\mathbb{M})^\Delta = \dim \left[ F_J(J, \mathbf{I}; \frac{N}{xN}) \right]^\Delta = \dim \frac{N}{xN : I^\infty}$$

by Note 2.1 and  $\tilde{e}(\mathbb{M}/\bar{x}\mathbb{M}) = \tilde{e}\left(F_J(J, \mathbf{I}; \frac{N}{xN})\right)$ . We get (ii). Since  $\bar{I}_i = \mathbb{S}_i$ , (iii) is obvious. ■

## 5. Recursion formulas for multiplicities of graded modules

This section gives the recursion formulas for the sum of all the mixed multiplicities of multi-graded modules. And as an application, we get the recursion formulas for the multiplicity of multi-graded Rees modules. Recall that  $\tilde{e}(M)$  denotes the sum of all the mixed multiplicities of  $M$ , i.e.,

$$\tilde{e}(M) = \sum_{|\mathbf{k}|=\ell-1} e(M; \mathbf{k});$$

$$S_i = S_{(0, \dots, \underbrace{1}_i, \dots, 0)} \text{ for all } i = 1, \dots, d;$$

$$S_{\hat{i}} = \bigoplus_{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d \geq 0; n_i=0} S_{(n_1, \dots, n_d)} \text{ and } M_{\hat{i}} = S_{\hat{i}} M_{(0, \dots, 0)}.$$

We have the following comment.

**Remark 5.1.** For any  $m \geq 0$ ,  $S_i^m M_{\hat{i}}$  is a finitely generated  $\mathbb{N}^{d-1}$ -graded  $S_{\hat{i}}$ -module. Since  $0 : S_i^u M_{\hat{i}} = 0 : S_i^v M_{\hat{i}}$  for all  $u, v \gg 0$ , there exists  $h$  such that  $\dim \text{Supp}_{++} S_i^u M_{\hat{i}} = \dim \text{Supp}_{++} S_i^v M_{\hat{i}}$  for all  $u, v \geq h$ . Hence by [24, Remark 3.1],  $\dim_{S_{\hat{i}}^\Delta} [S_i^u M_{\hat{i}}]^\Delta = \dim_{S_{\hat{i}}^\Delta} [S_i^v M_{\hat{i}}]^\Delta$  for all  $u, v \geq h$ .

The main result of this section is the following theorem.

**Theorem 5.2.** *Let  $S$  be a finitely generated standard  $\mathbb{N}^d$ -graded algebra over an artinian local ring  $A$  and let  $M$  be a finitely generated  $\mathbb{N}^d$ -graded  $S$ -module such that  $M = SM_{(0,\dots,0)}$ . Set  $\dim_{S^\Delta} M^\Delta = \ell$ . Assume that  $e(M; k_1, \dots, k_d) \neq 0$  and  $k_i > 0$ . Let  $x \in S_i$  be an  $S_{++}$ -filter-regular element with respect to  $M$ . Then the following statements hold.*

- (i)  $\tilde{e}\left(\frac{M}{xM}\right) = \sum_{|\mathbf{h}|=\ell-1; h_i>0} e(M; \mathbf{h})$ .
- (ii)  $\sum_{|\mathbf{h}|=\ell-1; h_i=0} e(M; \mathbf{h}) \neq 0$  if and only if  $\dim_{S_i^\Delta} [S_i^v M_i]^\Delta = \ell$  for some  $v \gg 0$ . In this case,  $\tilde{e}(S_i^v M_i) = \sum_{|\mathbf{h}|=\ell-1; h_i=0} e(M; \mathbf{h})$  for all  $v \gg 0$ .
- (iii) If  $\dim_{S_i^\Delta} [S_i^v M_i]^\Delta = \ell$  for some  $v \gg 0$  then  $\tilde{e}(M) = \tilde{e}\left(\frac{M}{xM}\right) + \tilde{e}(S_i^v M_i)$  for all  $v \gg 0$ .
- (iv) If  $\dim_{S_i^\Delta} [S_i^v M_i]^\Delta < \ell$  for some  $v \gg 0$  then  $\tilde{e}(M) = \tilde{e}\left(\frac{M}{xM}\right)$  for all  $v \gg 0$ .

**Proof.** Since  $x \in S_i$  is an  $S_{++}$ -filter-regular element with respect to  $M$ , we have

$$\ell_A\left[\left(\frac{M}{xM}\right)_{(n_1,\dots,n_d)}\right] = \ell_A[M_{(n_1,\dots,n_d)}] - \ell_A[M_{(n_1,\dots,n_i-1,\dots,n_d)}] \quad (7)$$

for all large  $n_1, \dots, n_d$  by [24, Remark 2.6]. Denote by  $P(n_1, \dots, n_i, \dots, n_d)$  the polynomial of  $\ell_A[M_{(n_1,\dots,n_d)}]$  and  $Q(n_1, \dots, n_d)$  the polynomial of  $\ell_A\left[\left(\frac{M}{xM}\right)_{(n_1,\dots,n_d)}\right]$ , from (7) we have

$$Q(n_1, \dots, n_d) = P(n_1, \dots, n_i, \dots, n_d) - P(n_1, \dots, n_i - 1, \dots, n_d). \quad (8)$$

Since  $e(M; k_1, \dots, k_d) \neq 0$  and  $k_i > 0$ , by (8) we get  $\deg Q = \deg P - 1$  and

$$e(M; h_1, \dots, h_d) = e\left(\frac{M}{xM}; h_1, \dots, h_i - 1, \dots, h_d\right) \text{ for all } h_i > 0. \quad (9)$$

By (9),

$$\sum_{|\mathbf{h}|=\ell-1; h_i>0} e(M; \mathbf{h}) = \sum_{|\mathbf{h}|=\ell-1; h_i>0} e\left(\frac{M}{xM}; h_1, \dots, h_i - 1, \dots, h_d\right).$$

Since  $\tilde{e}\left(\frac{M}{xM}\right) = \sum_{|\mathbf{h}|=\ell-1; h_i>0} e\left(\frac{M}{xM}; h_1, \dots, h_i - 1, \dots, h_d\right)$ ,

$$\tilde{e}\left(\frac{M}{xM}\right) = \sum_{|\mathbf{h}|=\ell-1; h_i>0} e(M; \mathbf{h}).$$

We have (i). Remember that

$$\tilde{e}(M) = \sum_{|\mathbf{h}|=\ell-1} e(M; \mathbf{h}) = \sum_{|\mathbf{h}|=\ell-1; h_i>0} e(M; \mathbf{h}) + \sum_{|\mathbf{h}|=\ell-1; h_i=0} e(M; \mathbf{h}).$$

Thus,

$$\tilde{e}(M) = \tilde{e}\left(\frac{M}{xM}\right) + \sum_{|\mathbf{h}|=\ell-1; h_i=0} e(M; \mathbf{h}). \quad (10)$$

Now, we prove (ii). Choose  $v \gg 0$  such that

$$P(n_1, \dots, n_d) = \ell_A[M_{(n_1, \dots, n_d)}]$$

for all  $n_1, \dots, n_d \geq v$ . Then  $P(n_1, \dots, v, \dots, n_d) = \ell_A[M_{(n_1, \dots, v, \dots, n_d)}]$  for all

$$n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d \geq v \text{ and } n_i = v.$$

Note that

$$\ell_A[M_{(n_1, \dots, v, \dots, n_d)}] = \ell_A[S_i^v M_{\hat{i}(n_1, \dots, 0, \dots, n_d)}]$$

and the terms of total degree  $\ell - 1$  in the polynomial

$$P(n_1, \dots, v, \dots, n_d) = \ell_A[S_i^v M_{\hat{i}(n_1, \dots, 0, \dots, n_d)}]$$

have the form

$$\sum_{h_1 + \dots + 0 + \dots + h_d = \ell-1} e(M; h_1, \dots, 0, \dots, h_d) \frac{n_1^{h_1} \dots v^0 \dots n_d^{h_d}}{h_1! \dots 0! \dots h_d!}.$$

This follows that  $\sum_{|\mathbf{h}|=\ell-1; h_i=0} e(M; \mathbf{h}) \neq 0$  if and only if  $\dim_{S_i^\Delta} [S_i^v M_{\hat{i}}]^\Delta = \ell$  for some  $v \gg 0$ . In this case,

$$e(M; h_1, \dots, h_{i-1}, 0, h_{i+1}, \dots, h_d) = e(S_i^v M_{\hat{i}}; h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_d)$$

for all  $v \gg 0$  by Remark 5.1. Therefore  $\tilde{e}(S_i^v M_{\hat{i}}) = \sum_{|\mathbf{h}|=\ell-1; h_i=0} e(M; \mathbf{h})$  for all  $v \gg 0$ . (ii) is proved. By (10) and (ii) we immediately get (iii) and (iv). ■

We now will discuss how particular cases of Theorem 5.2 can be treated.

Remember that if the multiplicities of  $M$ ;  $\frac{M}{xM}$ ;  $S_i^v M_{\hat{i}}$  are expressed as the sums of all the mixed multiplicities of  $M$ ;  $\frac{M}{xM}$ ;  $S_i^v M_{\hat{i}}$ , respectively then

$$e(M) = \tilde{e}(M); \quad e\left(\frac{M}{xM}\right) = \tilde{e}\left(\frac{M}{xM}\right); \quad e(S_i^v M_{\hat{i}}) = \tilde{e}(S_i^v M_{\hat{i}}).$$

Hence as an immediate consequence of Theorem 5.2, we have the following result.

**Corollary 5.3.** *Set  $\dim_{S^\Delta} M^\Delta = \ell$ . Assume that  $\ell > 1$ . Then we have:*

- (i) *If  $e(M)$  is the sum of all the mixed multiplicities of  $M$  and  $\dim_{S_i^\Delta} [S_i^v M_i]^\Delta = \ell$  for some  $v \gg 0$  then  $e(M) = \tilde{e}\left(\frac{M}{xM}\right) + \tilde{e}(S_i^v M_i)$  for all  $v \gg 0$ .*
- (ii) *If  $e\left(\frac{M}{xM}\right)$  is the sum of all the mixed multiplicities of  $\frac{M}{xM}$  then*

$$e\left(\frac{M}{xM}\right) = \sum_{|\mathbf{h}| = \ell-1; h_i > 0} e(M; \mathbf{h}).$$

- (iii) *If  $\dim_{S_i^\Delta} [S_i^v M_i]^\Delta = \ell$  and  $e(S_i^v M_i)$  is the sum of all the mixed multiplicities of  $S_i^v M_i$  for some  $v \gg 0$  then*

$$e(S_i^v M_i) = \sum_{|\mathbf{h}| = \ell-1; h_i = 0} e(M; \mathbf{h}) \text{ for all } v \gg 0.$$

Remember that  $\mathbb{S} = F_J(J, \mathbf{I}; R)$  and  $\mathbb{M} = F_J(J, \mathbf{I}; N)$ ;  $I = I_1 \cdots I_d$  is not contained in  $\sqrt{\text{Ann} N}$ ;  $\dim \frac{N}{0_N : I^\infty} = q$ . For any  $i = 1, \dots, d$ , set

$$\mathfrak{R}(\mathbf{I}_i; N) = \mathfrak{R}(I_1, \dots, I_{i-1}, I_{i+1}, \dots, I_d; N).$$

Recall that by Proposition 4.5(iii),

$$\mathbb{M}_i \cong F_J(J, I_1, \dots, I_{i-1}, I_{i+1}, \dots, I_d; N).$$

Upon simple computation, we get

$$\mathbb{S}_i^v \mathbb{M}_i \cong F_J(J, I_1, \dots, I_{i-1}, I_{i+1}, \dots, I_d; I_i^v N).$$

Set

$$\overline{N} = \frac{N}{0_N : I^\infty} \quad \text{and} \quad \overline{\mathbb{M}} = F_J(J, \mathbf{I}; \overline{N}).$$

Then since  $\text{ht} \frac{I + \text{Ann} \overline{N}}{\text{Ann} \overline{N}} > 0$ , we have

$$\dim I_i^v \overline{N} = \dim \overline{N} > \dim \frac{\overline{N}}{I_i^v \overline{N}}$$

for any  $1 \leq i \leq d$  and for all  $v > 0$ . Hence from short exact sequences

$$0 \longrightarrow I_i^v \overline{N} \longrightarrow \overline{N} \longrightarrow \frac{\overline{N}}{I_i^v \overline{N}} \longrightarrow 0,$$

by Corollary 3.9(ii)(b) we get

$$e((J, \mathfrak{R}(\mathbf{I}_i; R)_+); \mathfrak{R}(\mathbf{I}_i; \overline{N})) = e((J, \mathfrak{R}(\mathbf{I}_i; R)_+); \mathfrak{R}(\mathbf{I}_i; I_i^v \overline{N})).$$

On the other hand  $\tilde{e}(\mathbb{S}_i^v \mathbb{M}_i) = e((J, \mathfrak{R}(\mathbf{I}_i; R)_+); \mathfrak{R}(\mathbf{I}_i; I_i^v \overline{N}))$  by Corollary 2.6. Hence

$$\tilde{e}(\mathbb{S}_i^v \mathbb{M}_i) = e((J, \mathfrak{R}(\mathbf{I}_i; R)_+); \mathfrak{R}(\mathbf{I}_i; \overline{N})).$$

This fact yields:

**Note 5.4.**  $\tilde{e}(\mathbb{S}_i^v \mathbb{M}_i) = e((J, \mathfrak{R}(\mathbf{I}_i; R)_+); \mathfrak{R}(\mathbf{I}_i; \overline{N}))$ .

Put  $\mathfrak{J} = (J, \mathfrak{R}(\mathbf{I}; R)_+)$  and  $\mathfrak{J}_i = (J, \mathfrak{R}(\mathbf{I}_i; R)_+)$ . Then as a consequence of Theorem 5.2 and Proposition 4.5 we obtain the following results.

**Theorem 5.5.** *Assume that  $e(J^{[k_0+1]}, \mathbf{I}^{[k]}; N) \neq 0$  and  $k_i > 0$ . Let  $x \in I_i$  be a weak-(FC)-element of  $N$  with respect to  $(J, I_1, \dots, I_d)$ . Then*

$$(i) \quad e\left(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \frac{N}{xN : I^\infty})\right) = \sum_{h_0 + |\mathbf{h}| = q-1; h_i > 0} e(J^{[h_0+1]}, \mathbf{I}^{[\mathbf{h}]; N}).$$

$$(ii) \quad e\left(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \frac{N}{0_N : I^\infty})\right) = e\left(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \frac{N}{xN : I^\infty})\right) + e\left(\mathfrak{J}_i; \mathfrak{R}(\mathbf{I}_i; \frac{N}{0_N : I^\infty})\right).$$

$$(iii) \quad e\left(\mathfrak{R}(\mathbf{I}; \frac{N}{0_N : I^\infty})\right) = e\left(\mathfrak{R}(\mathbf{I}; \frac{N}{xN : I^\infty})\right) + e\left(\mathfrak{R}(\mathbf{I}_i; \frac{N}{0_N : I^\infty})\right).$$

**Proof.** Denote by  $\bar{x}$  the image of  $x$  in  $\mathbb{S}_i$ . Since  $x \in I_i$  is a weak-(FC)-element of  $N$  with respect to  $(J, I_1, \dots, I_d)$ ,  $\bar{x}$  is an  $\mathbb{S}_{++}$ -filter-regular element with respect to  $\mathbb{M}$  by Proposition 4.5(i). By Proposition 4.5(ii),  $\tilde{e}(\mathbb{M}/\bar{x}\mathbb{M}) = \tilde{e}\left(F(J, \mathbf{I}; \frac{N}{xN})\right)$ . Hence

$$\begin{aligned} \tilde{e}(\mathbb{M}/\bar{x}\mathbb{M}) &= e\left(F_J(J, \mathbf{I}; \frac{N}{xN : I^\infty})\right) \\ &= e\left((J, \mathfrak{R}(\mathbf{I}; R)_+); \mathfrak{R}(\mathbf{I}; \frac{N}{xN : I^\infty})\right) \\ &= e\left(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \frac{N}{xN : I^\infty})\right) \end{aligned}$$

by Corollary 2.6. Thus, we get (i) by Theorem 5.2(i). Now, since

$$\tilde{e}(\mathbb{M}) = e\left(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \frac{N}{0_N : I^\infty})\right) \text{ and } \tilde{e}(\mathbb{M}/\bar{x}\mathbb{M}) = e\left(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \frac{N}{xN : I^\infty})\right)$$

by Corollary 2.6, and

$$\tilde{e}(\mathbb{S}_i^v \mathbb{M}_{\hat{i}}) = e\left((J, \mathfrak{R}(\mathbf{I}_{\hat{i}}; R)_+); \mathfrak{R}(\mathbf{I}_{\hat{i}}; \overline{N})\right) = e\left(\mathfrak{J}_{\hat{i}}; \mathfrak{R}(\mathbf{I}_{\hat{i}}; \frac{N}{0_N : I^\infty})\right)$$

by Note 5.4, we have (ii) by Theorem 5.2(iii). Choose  $J = \mathbf{n}$ , we get (iii) by (ii). ■

Remember that if  $\text{ht } \frac{I + \text{Ann}N}{\text{Ann}N} > 0$ , then  $e\left(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \frac{N}{0_N : I^\infty})\right) = e(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; N))$  by Remark 2.7. Hence as an immediate consequence of Theorem 5.5, we have the following result.

**Corollary 5.6.** *Assume that  $\text{ht } \frac{I + \text{Ann}N}{\text{Ann}N} > 0$ ;  $e(J^{[k_0+1]}, \mathbf{I}^{[k]}; N) \neq 0$  and  $k_i > 0$ . Let  $x \in I_i$  be a weak-(FC)-element of  $N$  with respect to  $(J, I_1, \dots, I_d)$ . Then*

$$(i) \ e\left(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \frac{N}{xN : I^\infty})\right) = \sum_{h_0 + |\mathbf{h}| = q-1; h_i > 0} e(J^{[h_0+1]}, \mathbf{I}^{[\mathbf{h}]}; N).$$

$$(ii) \ e(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; N)) = e\left(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \frac{N}{xN : I^\infty})\right) + e(\mathfrak{J}_{\hat{i}}; \mathfrak{R}(\mathbf{I}_{\hat{i}}; N)).$$

$$(iii) \ e(\mathfrak{R}(\mathbf{I}; N)) = e\left(\mathfrak{R}(\mathbf{I}; \frac{N}{xN : I^\infty})\right) + e(\mathfrak{R}(\mathbf{I}_{\hat{i}}; N)).$$

Suppose that  $e(J^{[k_0+1]}, \mathbf{I}^{[k]}; N) \neq 0$  and  $x_1, \dots, x_p$  ( $p \leq k_i$ ) is a weak-(FC)-sequence in  $I_i$ . By Theorem 5.5 and by induction on  $p$ , we get the following corollary.

**Corollary 5.7.** *Let  $e(J^{[k_0+1]}, \mathbf{I}^{[k]}; N) \neq 0$  and  $x_1, \dots, x_p \in I_i$  ( $p \leq k_i$ ) be a weak-(FC)-sequence of  $N$  with respect to  $(J, I_1, \dots, I_d)$ . Then*

$$\begin{aligned} e\left(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \frac{N}{0_N : \mathfrak{J}^\infty})\right) &= e\left(\mathfrak{J}; \mathfrak{R}(\mathbf{I}; \frac{N}{(x_1, \dots, x_p)N : I^\infty})\right) \\ &+ \sum_{j=0}^{p-1} e\left(\mathfrak{J}_{\hat{i}}; \mathfrak{R}(\mathbf{I}_{\hat{i}}; \frac{N}{(x_1, \dots, x_j)N : I^\infty})\right). \end{aligned}$$

In particular, if  $d = 1$  then  $I = I_1$ . Put  $p = \max\{i \mid e(J^{[q-i]}, I^{[i]}; N) \neq 0\}$  and assume that  $x_1, \dots, x_p$  is a weak-(FC)-sequence of  $N$  with respect to  $(J, I)$ . Then by



[21, 22](see [11, Proposition 3.3(iii) and Theorem 3.4(iii)]),  $x_1, \dots, x_p$  is a maximal (FC)-sequence of  $N$  with respect to  $(J, I)$ . By Corollary 5.7,

$$e\left(\mathfrak{J}; \mathfrak{R}(I; \frac{N}{0 : I^\infty})\right) = e\left(\mathfrak{J}; \mathfrak{R}(I; \frac{N}{(x_1, \dots, x_p)N : I^\infty})\right) + \sum_{i=0}^{p-1} e\left(J; \frac{N}{(x_1, \dots, x_i)N : I^\infty}\right).$$

Since  $p$  is maximal,  $e(J^{[q-i]}, I^{[i]}; N) \neq 0$  if and only if  $0 \leq i \leq p$  by [21]. Consequently by [21](see [11, Proposition 3.3(i)]),

$$e(J^{[q-p-i]}, I^{[p+i]}; N) = e\left(J^{[q-p-i]}, I^{[i]}; \frac{N}{(x_1, \dots, x_p)N}\right) \neq 0$$

if and only if  $i = 0$ . Therefore by Corollary 2.5(ii),

$$e\left(\mathfrak{J}; \mathfrak{R}(I; \frac{N}{(x_1, \dots, x_p)N : I^\infty})\right) = e\left(J^{[q-p]}, I^{[0]}; \frac{N}{(x_1, \dots, x_p)N}\right).$$

On the other hand  $e\left(J^{[q-p]}, I^{[0]}; \frac{N}{(x_1, \dots, x_p)N}\right) = e\left(J; \frac{N}{(x_1, \dots, x_p)N : I^\infty}\right)$  by [11, Lemma 3.2]. Hence  $e\left(\mathfrak{R}(I; \frac{N}{(x_1, \dots, x_p)N : I^\infty})\right) = e\left(J; \frac{N}{(x_1, \dots, x_p)N : I^\infty}\right)$ . Thus,

$$e\left(\mathfrak{J}; \mathfrak{R}(I; \frac{N}{0_N : I^\infty})\right) = \sum_{j=0}^p e\left(J; \frac{N}{(x_1, \dots, x_j)N : I^\infty}\right).$$

Then we have the following corollary.

**Corollary 5.8.**  $e\left(\mathfrak{J}; \mathfrak{R}(I; \frac{N}{0_N : I^\infty})\right) = \sum_{j=0}^p e\left(J; \frac{N}{(x_1, \dots, x_j)N : I^\infty}\right).$

In the case that

$$\text{ht} \frac{I + \text{Ann}N}{\text{Ann}N} > 0, \quad e\left(\mathfrak{J}; \mathfrak{R}(I; \frac{N}{0_N : I^\infty})\right) = e(\mathfrak{J}; \mathfrak{R}(I; N))$$

by Remark 2.7. We get the following result which is proved by [11].

**Corollary 5.9** [11, Theorem 4.2]. *If  $\text{ht} \frac{I + \text{Ann}N}{\text{Ann}N} > 0$ , then*

$$e(\mathfrak{J}; \mathfrak{R}(I; N)) = \sum_{j=0}^p e\left(J; \frac{N}{(x_1, \dots, x_j)N : I^\infty}\right).$$

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